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VANISHING COHOMOLOGY THEOREMS AND STABILITY OF COMPLEX ANALYTIC FOLIATIONS

BY
JOAN GIRBAU

ABSTRACT

In this paper we deal with a complex analytic foliation of a compact complex manifold endowed with a bundle-like metric and give a transversally holomorphic rigidity theorem (Theorem 9.1) for these foliations, depending on curvature conditions. We give some examples for which we study holomorphic rigidity. The classical vanishing theorems of Nakano, Griffiths and Le Potier are the main tools we use to prove our results.

Introduction

The role of the vanishing theorems for the cohomology of complex analytic vector bundles is well known in the theory of complex manifolds. It is therefore to be supposed that analogous theorems might also prove interesting for complex analytic foliations. I. Vaisman formulated in [22] a vanishing theorem implying a rigidity result for some complex foliate structures, but, unfortunately, there was a sign error in its proof. In actual fact, the sign in formula (5.13) of [22] was wrong, and because of this, proposition 4 and the rigidity theorem on page 128 do not hold. I. Vaisman had the opportunity to announce this error in the review of the mentioned paper in Mathematical Reviews. (See Example 9.1 of the present paper and the Remark that follows it in connexion with the results of [22] that remain valid.)

We actually give a new vanishing cohomology result (Theorem 8.1) for foliate vector bundles implying a rigidity theorem for complex foliations (Theorem 9.1). We also give some examples in which Theorem 9.1 proves to be useful. Theorem 3.1 constitutes the main tool which makes it possible to apply the classical vanishing theorems of Nakano, Griffiths and Le Potier to the foliate case.

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Theorem 3.1 is based on a generalization [9] of a rigidity theorem of R. S. Hamilton [14] in connexion with compact Hausdorff foliations.

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1. Review of compact Hausdorff foliations

This section summarizes material presented in detail in [10] and [9].

Let M be a C^∞ manifold of dimension $n+m$ with a C^∞ foliation $\mathcal{F}$ of codimension n. Denote by $B = M/\mathcal{F}$ the quotient space obtained by identifying each leaf of $\mathcal{F}$ to a point, endowed with the quotient topology. We say that $\mathcal{F}$ is compact Hausdorff if the leaves are compact and B is Hausdorff.

EXAMPLE 1.1. Let D be the open unit ball in $\mathbf{R}^n$. Let G be a finite subgroup of $O(n)$. Let L be a compact manifold. Suppose that there is a C^∞ free action of G on L, on the right. Define an action of G on $L \times D$ by $g(p,x) = (pg^{-1}, gx)$. The quotient space $L \times_G D$ of $L \times D$ by this action can be endowed with a structure of C^∞ manifold in a natural way (by using the fact that the action of G on L is free). Take the foliation on $L \times D$ whose leaves are $L \times \{\text{point}\}$. This foliation is preserved by the action of G. So we have a foliation induced on $L \times_G D$. This foliation is compact Hausdorff.

This example is interesting by virtue of the following theorem due to Reeb, Ehresmann, Haefliger and Epstein [7] (see also [13]), that asserts that all compact Hausdorff foliations are locally like the one of this example.

THEOREM 1.1. [7] *If $\mathcal{F}$ is compact Hausdorff, then there is a "generic leaf" L such that there is a dense open subset of M where the leaves are all diffeomorphic to L. Moreover, given a leaf L_0, there is*: (a) *a finite subgroup G of $O(n)$*; (b) *a free C^∞ action of G on L, on the right*; (c) *an open neighborhood V of L_0 and a C^∞ diffeomorphism $\Phi: L \times_G D \to V$ which preserves the leaves if one takes on $L \times_G D$ the foliation of Example* 1.1.

DEFINITION 1.1. Let $L \times_G D$ be the foliate manifold of Example 1.1. Let $\eta: G \to C^\infty(D, \mathrm{GL}(r, \mathbf{R}))$ be a mapping such that $\eta(gg')(x) = \eta(g)(g'x) \circ \eta(g')(x)\ \forall g, g' \in G$ and $\forall x \in D$. Define an action of G on $L \times D \times \mathbf{R}^r$ by $g(p, x, v) = (pg^{-1}, gx, \eta(g)(x)v)$. One can prove easily that $(L \times D \times \mathbf{R}^r)/G \to L \times_G D$ (natural projection) constitutes a vector bundle of fibre $\mathbf{R}^r$ (see [10]). Such a fibre bundle on $L \times_G D$ will be called *allowable*.

DEFINITION 1.2. Suppose that the foliation $\mathcal{F}$ over M is compact Hausdorff. A C^∞ vector bundle $E \to M$ will be called *allowable* if, given a leaf L_0, there is a finite subgroup G of $O(n)$, a free action of G on L, a neighborhood V of L_0, and a diffeomorphism $\Phi: L \times_G D \to V$ verifying the properties of Theorem 1.1 such that the pull-back of $E \mid V$ by Φ is allowable on $L \times_G D$ (Definition 1.1).

The transversal bundle of $\mathcal{F}$ and the trivial bundle are allowable. From the functorial character of the definition it follows that if E and F are allowable, then $E \otimes F$, $E \oplus F$, $\mathrm{Hom}(E, F)$, $\Lambda^p E$, E^*, etc. are also allowable. Recall now the following

DEFINITION 1.3. A vector bundle $E \to M$ is called foliate if there is an open cover $\mathcal{U} = \{W\}$ of M by flat local charts $(W, x^1, \cdots x^n, x^{n+1} \cdots x^{n+m})$ and a trivialization of $E \mid W$ over each $W \in \mathcal{U}$, given by a basis of sections $s_1 \cdots s_r$, in such a way that the transition functions are constant along the leaves (that is, they depend only on $x^1 \cdots x^n$). If E is such, we have the notion of a base-like section. A section γ of E is called *base-like* if it has an expression $\gamma = \Sigma \gamma^A s_A$, $A = 1 \cdots r$, over each $W \in \mathcal{U}$, where the functions γ^A depend only upon $x^1 \cdots x^n$.

It is easy to see that an allowable vector bundle is always foliate [10].

We need the following

THEOREM 1.2. [9]. *Suppose that the foliation $\mathcal{F}$ of M is compact Hausdorff. Let $E \to M$ be an allowable vector bundle. Let $\mathfrak{A}(E)$ be the sheaf of germs of C^∞ base-like local sections of E. Let L be the generic leaf of $\mathcal{F}$. If $b_1(L) = 0$, then $H^1(M, \mathfrak{A}(E)) = 0$.*

REMARK. We can repeat this section assuming that E is a vector bundle with fibre $\mathbf{C}^r$ (instead of $\mathbf{R}^r$).

2. Review of complex analytic foliations

From now on M will be a compact complex manifold of complex dimension $n + m$ endowed with a complex analytic foliation $\mathcal{F}$ of complex codimension n whose leaves are closed subsets in M. We suppose that $\mathcal{F}$ is defined by an adapted atlas $\{(U_\alpha, z^a_\alpha, z^u_\alpha)\}$ (index convention: $a, b \cdots = 1 \cdots n$; $u, v \cdots = n + 1 \cdots n + m$) where $z^a_\alpha =$ constant defines the leaves. We suppose that M is endowed with a Hermitian bundle-like metric g. We shall denote by $*$ the Hodge star operator corresponding to g and by $\bar{*}$ the operator $\bar{*}\varphi = \overline{*\varphi}$ where $\overline{*\varphi}$ means the complex conjugate of $*\varphi$. We shall have a Hermitian scalar product defined in the space of differential forms by $\langle \varphi, \psi \rangle = \int_M \varphi \wedge \bar{*}\psi$ and the operator $\delta = -\bar{*}d\bar{*}$ such that $\langle d\varphi, \psi \rangle = \langle \varphi, \delta\psi \rangle$. We shall have decompositions $d = d' + d''$, $\delta = \delta' + \delta''$ in accordance with complex types. We shall denote by $D^{p,q}$ the space of the forms φ which have the following local expression:

$$\varphi = \frac{1}{p!}\frac{1}{q!}\varphi_{a_1\cdots a_p\bar{b}_1\cdots\bar{b}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge dz^{\bar{b}_1} \cdots \wedge dz^{\bar{b}_q} \tag{2.1}$$

where the coefficients $\varphi_{a_1\cdots a_p\bar{b}_1\cdots\bar{b}_q}$ depend only on z^a, $\overline{z^a}$ (recall that $a, b \cdots = 1 \cdots n$). The elements of $D^{p,q}$ will be called base-like (p,q)-forms.

At each point $x_0 \in M$ we can define algebraically a Hodge star operator $\bar{*}_b : D^{p,q}_{x_0} \to D^{n-p,n-q}_{x_0}$ (we use b as initial of "base-like"), using only the transversal part $(g_{a\bar{b}})$ of the metric g, in the way that we are going to explain. We can always suppose that the coordinates z^a have been taken such that the matrix $(g_{a\bar{b}})$ is the identity at x_0. Let $A_p = \{a_1 \cdots a_p\}$ be a set of indices $a_1 < \cdots < a_p$, $0 \leqq a_i \leqq n$. Denote by $A'_p = \{1 \cdots n\} - A_p$ ordered by $<$. We define $\bar{*}_b$ at x_0 to be

$$\bar{*}_b(f dz^{A_p} \wedge dz^{\bar{B}_q}) = (-1)^{q(n-p)}\varepsilon(A_p, A'_p)\varepsilon(B_q, B'_q)\bar{f} dz^{A'_p} \wedge dz^{\overline{B'_q}}$$

where f is any complex number. Define δ_b on $D^{p,q}$ by $\delta_b = -\bar{*}_b d\bar{*}_b$. We recall that by virtue of a lemma of [10, chap. 4] we have $\delta_b\varphi = \delta\varphi$ if $\varphi \in D^{p,q}$.

Let $E \to M$ be a complex analytic foliate vector bundle (here complex analytic foliate means that it is possible to find a cover $\mathcal{U} = \{W\}$, analogous to the one of Definition 1.3, by flat local charts, (W, z^a, z^u), and trivializations over each W such that the transition functions depend only on z^a). We suppose that E is endowed with a foliate Hermitian metric h (here foliate means that h is locally given by a Hermitian matrix (h_{AB}) depending only on z^a and $\overline{z^a}$) (index convention: $A, B \cdots = 1 \cdots r$). We shall use the associate Hermitian connexion locally given by $\omega_A^B = (d'h_{AC})h^{CB}$. We can define a Hermitian product in the space of E-valued differential forms by $\langle \varphi, \psi \rangle = \int_M h_{AB}\varphi^A \wedge \bar{*}\psi^B$. (Recall that if φ is an E-valued differential form, then φ has the local expression $\varphi = \varphi^A \otimes s_A$, where φ^A are ordinary forms and $\{s_A\}$ is the basis of sections given by the trivialization we have taken.)

We define the operators $d'_E, d''_E, \delta'_E, \delta''_E$ in the space of E-valued forms by the local expressions

$$(d'_E\varphi)^A = d'\varphi^A + \omega_B^A \wedge \varphi^B, \qquad (d''_E\varphi)^A = d''\varphi^A,$$

$$(\delta'_E\varphi)^A = \delta'\varphi^A, \qquad (\delta''_E\varphi)^A = \delta''\varphi^A - \bar{*}e(\theta)\bar{*}\varphi^A,$$

where $e(\theta)$ means the exterior product by the matrix $\theta = (\theta_A^B)$, $\theta_A^B = h^{BC}d''h_{CA}$.

One verifies that d'_E (resp. d''_E) and δ'_E (resp. δ''_E) are adjoints with respect to the scalar product $\langle \ , \ \rangle$.

We shall denote by $D^{p,q}(E)$ the space of C^∞ base-like E-valued (p,q)-forms. Define the operators $(\delta'_E)_b$ and $(\delta''_E)_b$ on $D^{p,q}(E)$ by $(\delta'_E)_b = \delta'_b$ and $(\delta''_E)_b = \delta''_b - \tilde{*}_b e(\theta)\tilde{*}_b$ on each local chart. Since the metric h is foliate, $(\delta''_E)_b$ maps $D^{p,q}(E)$ into $D^{p,q-1}(E)$. One can prove that on $D^{p,q}(E)$ one has $(\delta''_E)_b = \delta''_E$ and $(\delta'_E)_b = \delta'_E$ (as in case E is trivial).

Let Δ''_E be the Laplace operator $d''_E\delta''_E + \delta''_E d''_E$. On $D^{p,q}(E)$ one has $\Delta''_E = d''_E(\delta''_E)_b + (\delta''_E)_b d''_E$. Hence Δ''_E maps $D^{p,q}(E)$ into itself. We shall denote by $H_b^{p,q}(M,E)$ the space of those $\varphi \in D^{p,q}(E)$ such that $\Delta''_E\varphi = 0$.

Recall that M is compact, endowed with a bundle-like metric, and that its leaves are closed. Then a known theorem (see [16]) asserts that the foliation is compact Hausdorff as a real C^∞ foliation over the $2(n+m)$-dimensional real manifold M.

We need the following

THEOREM 2.1 [10]. *Let $E \to M$ be a complex analytic foliate vector bundle endowed with a foliate Hermitian metric. Assume that $E \to M$ is allowable (Definition 1.2) over the $2(n+m)$-dimensional real manifold M. Let $H^q(D^{p,\cdot}(E), d''_E)$ be the cohomology of the complex*

$$\cdots \longrightarrow D^{p,i}(E) \xrightarrow{d''_E} D^{p,i+1}(E) \longrightarrow \cdots.$$

We have $H^q(D^{p,\cdot}(E), d''_E) \cong H_b^{p,q}(M,E)$.

3. Cohomology of base-like forms

Keep the notations and assumptions of the preceding section. Denote by $\Omega_b^p(E)$ the sheaf of germs of holomorphic local E-valued base-like $(p,0)$-forms. Let L be the generic leaf of $\mathcal{F}$. We are going to prove the following

THEOREM 3.1. *Let $E \to M$ be a complex analytic foliate vector bundle endowed with a foliate Hermitian metric h. Assume that $E \to M$ is allowable over the $2(n+m)$-dimensional real manifold M. If $b_1(L) = 0$, then $H^1(M, \Omega_b^p(E)) \cong H_b^{p,1}(M,E)$.*

PROOF. Here we borrow some ideas from [15]. Denote by τ the tangent bundle of type $(1,0)$ of the foliation. Denote by $T(M)$ the tangent bundle of type $(1,0)$ of M. Let $\nu = T(M)/\tau$ the transversal bundle. Given $X \in \Gamma(T(M))$, we shall denote by $\langle X\rangle$ its class in ν. We take a connexion ∇ of type $(1,0)$ in ν such that if $X \in \Gamma(\tau)$ and $\langle Y \rangle \in \Gamma(\nu)$ then $\nabla_X \langle Y \rangle = \langle [X, Y] \rangle$ (Bott [3]). We take the connexion in E introduced in the preceding section. These connexions induce a connexion in $\Lambda^p \nu^* \otimes E$. Denote by ξ the bundle $\bar{\nu} \oplus \tau \oplus \bar{\tau}$. Denote by $B^{p,q} = \Gamma(\Lambda^q \xi^* \otimes \Lambda^p \nu^* \otimes E)$. Define an operator $\hat{d} : B^{p,q} \to B^{p,q+1}$ in the following way. If $\sigma \in B^{p,q}$ and $X_0 \cdots X_q \in \Gamma(\xi)$, we define

$$(\hat{d}\sigma)(X_0 \cdots X_q) = \sum_{0 \leq i \leq q} (-1)^i \nabla_{X_i} \sigma(X_0 \cdots \hat{X}_i \cdots X_q)$$

$$+ \sum_{0 \leq i < j \leq q} (-1)^{i+j} \sigma([X_i, X_j], X_0 \cdots \hat{X}_i \cdots \hat{X}_j \cdots X_q).$$

Since E is foliate there is a cover $\mathcal{U}$ consisting of flat local charts (U, z^a, z^u) and trivializations of $E \mid U$ for any $U \in \mathcal{U}$ such that the transition functions depend only upon z^a. Let us find the expression of $\hat{d}$ on (U, z^a, z^u). Denote by x^i the coordinates $\overline{z^a}, z^u, \overline{z^u}$. If $\sigma \in B^{p,q}$, σ will have the local expression

$$\sigma = \frac{1}{p!} \frac{1}{q!} \sigma^A_{i_1 \cdots i_q a_1 \cdots a_p} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \wedge dz^{a_1} \wedge \cdots \wedge dz^{a_p} \otimes s_A .$$

Let X be one of the following vector fields over U, $X = \partial / \partial z^u$, $X = \partial / \partial \overline{z^u}$ or $X = \partial / \partial \overline{z^a}$. Observe that we have $\nabla_X dz^b = 0$ and $\nabla_X s_A = 0$ with respect to the connexions taken in ν^* and E. In fact, when $X = \partial / \partial \overline{z^a}$ or $X = \partial / \partial \overline{z^u}$ these identities are fulfilled since these connexions are of type (1, 0). When $X = \partial / \partial z^u$ we have

$$\nabla_{\partial / \partial z^u} \left\langle \frac{\partial}{\partial z^a} \right\rangle = \left\langle \left[\frac{\partial}{\partial z^u}, \frac{\partial}{\partial z^a} \right] \right\rangle = 0.$$

We also have $\nabla_{\partial / \partial z^u} s_A = 0$ since the connexion in E is given by $\omega^B_A = (d' h_{AC}) h^{CB}$ where h is foliate, that is, h_{AC} depends only on $z^a, \overline{z^a}$. From this we deduce that

$$\hat{d}\sigma = \frac{1}{p!} \frac{1}{q!} \left(\frac{\partial}{\partial x^i} (\sigma^A_{i_1 \cdots i_p a_1 \cdots a_p}) dx^i \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_q} \wedge dz^{a_1} \cdots dz^{a_q} \otimes s_A .$$

Hence $\hat{d}^2 = 0$. Denote by $A^{p,r,s} = \Gamma(\Lambda^r \overline{\nu^*} \otimes \Lambda^s (\tau^* \oplus \overline{\tau^*}) \otimes \Lambda^p \nu^* \otimes E)$. Obviously $B^{p,q} = \Sigma_{r+s=q} A^{p,r,s}$. The operator $\hat{d}$ maps $A^{p,r,s}$ into $A^{p,r+1,s} \oplus A^{p,r,s+1}$. Hence $\hat{d}$ has an obvious decomposition into two parts, $\hat{d} = d'' + d_f$. $\hat{d}^2 = 0$ implies $d''^2 = 0$, $d_f^2 = 0$ and $d'' d_f = -d_f d''$. Let $\mathfrak{A}^{p,r,s}$ be the sheaf of germs of local sections of $\Lambda^r \overline{\nu^*} \otimes \Lambda^s (\tau^* \oplus \overline{\tau^*}) \otimes \Lambda^p \nu^* \otimes E$. The sequence of sheafs

$$\cdots \longrightarrow \mathfrak{A}^{p,r,s} \xrightarrow{d_f} \mathfrak{A}^{p,r,s+1} \longrightarrow \cdots$$

is exact by virtue of lemma 2.4 of [15] (our operator d_f is analogous to the operator $\hat{d}$ of [15] in the real case considered there). Observe that

$$\mathrm{Ker}\{d_f : \mathfrak{A}^{p,0,0} \to \mathfrak{A}^{p,0,1}\} = \mathfrak{A}^p(E),$$

where $\mathfrak{A}^p(E)$ is the sheaf of germs of C^∞ E-valued base-like local $(p,0)$-forms. We have then the resolution

$$0 \longrightarrow \mathfrak{A}^p(E) \longrightarrow \mathfrak{A}^{p,0,0} \xrightarrow{d_f} \mathfrak{A}^{p,0,1} \longrightarrow \cdots.$$

Since the sheaves $\mathfrak{A}^{p,0,s}$ are fine, $H^q(\{A^{p,0,\cdot}, d_f\}) = H^q(M, \mathfrak{A}^p(E))$. By virtue of Theorem 1.2 applied to the vector bundle $\wedge^p\nu \otimes E$ we have $H^1(M, \mathfrak{A}^p(E)) = 0$. Consider the double complex

$$\begin{array}{ccccc}
\uparrow & & \uparrow & & \\
A^{p,0,1} & \xrightarrow{d''} & A^{p,1,1} & \longrightarrow & \cdots \\
d_f \uparrow & & \uparrow & & \\
A^{p,0,0} & \xrightarrow{d''} & A^{p,1,0} & \longrightarrow & \cdots \\
i \uparrow & & \uparrow & & \\
D^{p,0}(E) & \xrightarrow{d''} & D^{p,1}(E) & \longrightarrow & \cdots
\end{array}$$

Observe that the kernel of $d_f : A^{p,r,0} \to A^{p,r,1}$ is $D^{p,r}(E)$. We are going to prove that $H^1(\{B^{p,\cdot}, \hat{d}\}) \cong H^1(\{D^{p,\cdot}(E), d''\})$. In fact, given $u = u^{p,0,1} + u^{p,1,0} \in B^{p,1}$ such that $\hat{d}u = 0$, we have $d_f u^{p,0,1} = 0$, $d''u^{p,0,1} = -d_f u^{p,1,0}$ and $d''u^{p,1,0} = 0$. Since $H^1(\{A^{p,0,\cdot}, d_f\}) = 0$ there exists $u^{p,0,0} \in A^{p,0,0}$ such that $d_f u^{p,0,0} = u^{p,0,1}$. We have $d_f(d''u^{p,0,0} - u^{p,1,0}) = -d''d_f u^{p,0,0} - u^{p,1,0} = -d''u^{p,0,1} - d_f u^{p,1,0}$. Hence $(d''u^{p,0,0} - u^{p,1,0}) \in D^{p,1}(E)$. This element is a cocycle of $D^{p,1}(E)$ since $d''(d''u^{p,0,0} - u^{p,1,0}) = -d''u^{p,1,0} = 0$. The map from $H^1(\{B^p, \hat{d}\})$ to $H^1(\{D^{p,\cdot}(E), d''\})$ given by class $u \to$ class of $(d''u^{p,0,0} - u^{p,1,0})$ is well defined and it is an isomorphism. (In fact, given a cocycle $a \in D^{p,1}(E)$, then $0 + a \in A^{p,0,1} \oplus A^{p,1,0} = B^{p,1}$ is a cocycle and the mapping: class $a \to$ class $(0 + a)$ is the inverse isomorphism.) As in [15] we have $H^1(\{B^p, \hat{d}\}) = H^1(M, \Omega_b^p(E))$. Hence $H^1(M, \Omega_b^p(E)) \cong H^1(\{D^{p,\cdot}(E), d''\})$. By virtue of Theorem 2.1 we have $H^1(M, \Omega_b^p(E)) \cong H_b^{p,1}(M, E)$.

4. Bundle-like pseudo-Kähler metrics

Let g be a bundle-like Hermitian metric on M locally given by

$$g = g_{a\bar{b}} dz^a \wedge \overline{dz^b} + g_{u\bar{v}} \theta^u \wedge \overline{\theta^v}, \tag{4.1}$$

$a, b = 1 \cdots n$; $u, v = n+1 \cdots n+m$, where $g_{a\bar{b}}$ depends only on $z^1 \cdots z^n$,

$\overline{z^1} \cdots \overline{z^n}$. We shall say that g is a bundle-like pseudo-Kähler metric if $d\omega' = 0$, where

$$\omega' = \sqrt{-1}\, g_{a\bar{b}} dz^a \wedge d\overline{z^b}.$$

(This expression gives a form globally defined.) We shall suppose from now on that g is such. Denote by L' the left exterior multiplication by ω'. As in section 2 denote by $D^{p,q}$ the space of C^∞ base-like (p,q)-forms. Define Λ' on $D^{p,q}$ by $\Lambda' = \tilde{*}_b^{-1} L' \tilde{*}_b$. The pseudo-Kähler condition implies $dL' = L'd$ on $D^{p,q}$. From this relation (using $*_b$) one also gets $\delta_b \Lambda' = \Lambda' \delta_b$. Using $\tilde{*}_b$ and the same reasoning as in [4] one can get (on $D^{p,q}$) the identity $\Lambda' d - d\Lambda' = -C^{-1}\delta_b C$. From this identity one obtains on $D^{p,q}$

$$(4.2) \qquad \begin{cases} \Lambda' d' - d'\Lambda' = \sqrt{-1}\,\delta_b'', \\ \Lambda' d'' - d''\Lambda' = -\sqrt{-1}\,\delta_b'. \end{cases}$$

Let $E \to M$ be a complex analytic foliate vector bundle endowed with a foliate Hermitian metric h. Let $D^{p,q}(E)$ be the space of C^∞ base-like E-valued forms of type (p,q). Let ω be the connexion form $\omega_A^B = (d'h_{AC})h^{CB}$. Let θ be the matrix form $\theta_A^B = h^{BC}(d''h_{CA})$. ($\omega$ and θ are only defined on each local chart.) One can prove by a componentwise computation that one has on $D^{p,q}(E)$

$$(4.3) \qquad \Lambda' e(\omega) - e(\omega)\Lambda' = -\sqrt{-1}\,\tilde{*}_b e(\theta) \tilde{*}_b = -\sqrt{-1}\,\tilde{*} e(\theta) \tilde{*}.$$

From (4.2) and (4.3) one gets on $D^{p,q}(E)$

$$(4.4) \qquad \begin{cases} \Lambda' d_E' - d_E'\Lambda' = \sqrt{-1}\,\delta_E'', \\ \Lambda' d_E'' - d_E''\Lambda' = -\sqrt{-1}\,\delta_E'. \end{cases}$$

One verifies easily

$$(4.5) \qquad d_E' d_E'' + d_E'' d_E' = e(\Omega)$$

where $\Omega_A^B = d''\omega_A^B$.

We shall also employ the operator $\tilde{\#}_b : D^{p,q}(E) \to D^{n-p,n-q}(E^*)$ defined on each local chart $(U_\alpha, z_\alpha^a, z_\alpha^u)$ by $(\tilde{\#}_b \varphi) | U_\alpha = (h_{AB})_\alpha \tilde{*}_b \varphi_\alpha^B$. One can see that $(h_{AB})_\alpha \tilde{*}_b \varphi_\alpha^B$ defines an E^*-valued global form.

5. A vanishing cohomology theorem for complex analytic foliate line bundles

Everything is now ready for the proof of the following generalized Nakano's result.

THEOREM 5.1. *Let M be a compact manifold of complex dimension $n+m$ endowed with a complex analytic foliation $\mathcal{F}$ of complex codimension n. Suppose that we are given a bundle-like pseudo-Kähler metric g on M. Let $E \to M$ be a complex analytic foliate line bundle with a foliate Hermitian metric h. Let Ω be the curvature form of the connexion associated to h. Let*

$$\Omega = \Omega_{a\bar{b}} dz^a \wedge \overline{dz^b} \qquad (a, b = 1 \cdots n)$$

be the local expression of Ω. Suppose that the matrix $(\Omega_{a\bar{b}})$ defines, at each point x, a negative definite Hermitian product in ν_x (ν = transversal bundle). Then there is a bundle-like pseudo-Kähler metric $\tilde{g}$ on M such that $H_b^{p,q}(M, E) = 0$ for $p+q<n$, where $H_b^{p,q}(M, E)$ are referred to $\tilde{g}$.

PROOF. Observe that Ω will be a global base-like form. Let g be the initial bundle-like pseudo-Kähler metric (whose local expression is (4.1)). By the hypotheses, the expression

$$\tilde{g} = -\Omega_{a\bar{b}} dz^a \overline{dz^b} + g_{u\bar{v}} \theta^u \overline{\theta^v}$$

will define another bundle-like pseudo-Kähler metric. Let $H_b^{p,q}(M, E)$ be the space of C^∞ base-like E-valued (p, q)-forms φ such that $\Delta_E''\varphi = 0$, where Δ_E'' is referred to $\tilde{g}$. Let φ be an element of $H_b^{p,q}(M, E)$. We shall have

$$\begin{aligned}
\langle d_E'\varphi, d_E'\varphi\rangle &= \langle \varphi, \delta_E' d_E'\varphi\rangle = -\sqrt{-1}\langle \varphi, (\Lambda' d_E'' - d_E''\Lambda') d_E'\varphi\rangle \\
&= -\sqrt{-1}\langle(\delta_E'' L' - L'\delta_E'')\varphi, d_E'\varphi\rangle = -\sqrt{-1}\langle \delta_E'' L'\varphi, d_E'\varphi\rangle \\
&= -\sqrt{-1}\langle L'\varphi, d_E'' d_E'\varphi\rangle = -\sqrt{-1}\langle L'\varphi, (d_E'' d_E' + d_E' d_E'')\varphi\rangle \\
&= -\sqrt{-1}\langle L'\varphi, e(\Omega)\varphi\rangle.
\end{aligned}$$

(Recall that $\langle\ ,\ \rangle$ is Hermitian.) But $L' = -\sqrt{-1}\, e(\Omega)$ by virtue of the choice of g. So $\langle d_E'\varphi, d_E'\varphi\rangle = -\langle L'\varphi, L'\varphi\rangle$. This implies $L'\varphi = 0$ and $d_E'\varphi = 0$. But it is well known that $L'\varphi = 0$ implies $\varphi = 0$ if $p+q<n$ (see [1]).

6. Le Potier's isomorphism

Consider again the complex foliate manifold M of the preceding section. Let $\pi : E \to M$ be a complex analytic foliate vector bundle with r-dimensional fibres. Let $\pi' : E^* \to M$ be its dual bundle. Denote by $p : P(E^*) \to M$ the projective bundle defined by the 1-dimensional subspaces of the fibres of E^*. (Each point u of $P(E^*)$ is a 1-dimensional subspace of $\pi'^{-1}p(u)$.) Since E is foliate we can take a cover $\mathcal{U}$ of M consisting of flat local charts (U, z^a, z^u) and a trivialization of $E^* \mid U$ for any $U \in \mathcal{U}$ such that the transition functions depend only on z^a. Then for any $U_\alpha \in \mathcal{U}$ one has the natural coordinates $(z^a_\alpha, z^u_\alpha, y^A_\alpha)$ $(A = 1 \cdots r)$ on $\pi'^{-1}(U_\alpha) \cong U_\alpha \times C^r$. Let $V_{A,\alpha}$ be the subset of $p^{-1}(U_\alpha)$ given by $y^A_\alpha \neq 0$. On $V_{A,\alpha}$ we have the local coordinates

$$(z^a_\alpha, z^u_\alpha, t^B_{A,\alpha} = y^B_\alpha / y^A_\alpha), \qquad B \neq A. \tag{6.1}$$

It is easy to see (since E is foliate) that (6.1) defines a flat coordinate system for a complex foliation on $P(E^*)$ whose leaves are defined by $z^a_\alpha = \text{constant}$, $t^B_{A,\alpha} = \text{constant}$. The codimension of this foliation is $n + r - 1$.

Let $p^*(E)$ be the pull-back of E by $p : P(E^*) \to M$. Let S be the subbundle of $p^*(E)$ consisting of the pairs (v_z, u_z) $(z \in M,\ v_z \in E^*_z,\ u_z \in E_z)$ such that $v_z(u_z) = 0$. Then the quotient bundle $Q(E) = p^*(E)/S$ is a complex analytic foliate line bundle on $P(E^*)$. Le Potier proved [19] (for $m = 0$) some cohomology isomorphisms. (See [20] for another proof.) It is easy to see that the proof in [20] holds in our situation. Precisely, one has the following

THEOREM 6.1. *For every complex foliate manifold M endowed with a complex analytic foliate vector bundle $\pi : E \to M$, there are the following isomorphisms:*

$$H^q(M, \Omega^p_b(E)) \cong H^q(P(E^*), \Omega^p_b(Q(E))), \tag{6.2}$$

if one takes the foliation on $P(E^*)$ given by (6.1).

Let h be a foliate Hermitian metric in E. Denote by h^* its induced metric in E^*. We shall use a tilda ($\sim$) to denote the passage to equivalence classes either in $P(E^*)$ or in $Q(E)$. We shall obtain a Hermitian metric H in $Q(E)$ by

$$H(\widetilde{(\tilde{y}_z, u_z)}, \widetilde{(\tilde{y}'_z, u'_z)}) = \frac{y_z(u_z) \cdot \overline{y'_z(u'_z)}}{h^*(y_z, y'_z)}, \tag{6.3}$$

which is easily seen to be independent of the chosen representatives and to be a foliate metric. Let γ be the curvature form (on $P(E^*)$) associated to this metric. We want to calculate γ at a fixed point of $P(E^*)$. We may suppose that this point belongs to the typical coordinate neighborhood $V_{r,\alpha}$ with non-homogeneous coordinates $(z^a, z^u, t^1 \cdots t^{r-1}, 1)$. Here our index convention will be: $A, B \cdots = 1 \cdots r$ with $t^r = 1$ and $P, Q \cdots = 1 \cdots r-1$. If a trivialization of E^* is chosen on some neighborhood of the fixed point such that h^* is the unit matrix and dh^* the zero matrix at this point, one obtains as in [12, page 202] the following expression for γ:

$$(6.4) \quad \begin{cases} \gamma_{a\bar{b}} = \dfrac{(\partial_a \partial_{\bar{b}} h^*_{AB}) t^A \overline{t^B}}{1 + \Sigma t^P \overline{t^P}}; \quad \gamma_{a\bar{P}} = 0, \\[2ex] \gamma_{P\bar{Q}} = \dfrac{\delta_{PQ}}{1 + \Sigma t^P \overline{t^P}} - \dfrac{t^P \overline{t^Q}}{(1 + \Sigma t^P \overline{t^P})^2}; \quad \gamma_{u\bar{v}} = \gamma_{u\bar{P}} = 0, \end{cases}$$

which is valid at the chosen point and for the chosen trivialization.

7. Transversal curvature, transversal Ricci tensor and canonical bundle

Let g be a bundle-like Hermitian metric on M locally given by (4.1). Let ν be the transversal bundle. By means of g, ν can be identified to the normal bundle. Let $\{dz^a, \theta^u\}$ be the local cobasis of (1,0)-forms which appears in the expression (4.1). Let $\{Z_a, \partial/\partial z^u\}$ be its dual basis. The matrix $(g_{a\bar{b}})$ which appears in (4.1) defines a Hermitian metric in the normal bundle ν by the formula $g(Z_a, \overline{Z_b}) = g_{a\bar{b}}$. Let ∇ be the unique connexion in ν of type (1,0) associated to this metric, given locally by $\omega_a^b = (d' g_{a\bar{c}}) g^{\bar{c}b}$. Let

$$R : \Gamma(T(M)) \times \Gamma(\overline{T(M)}) \times \Gamma(\nu) \rightarrow \Gamma(\nu)$$

be the curvature tensor of this connexion. One has $R(X, \bar{Y})Z = 0$ if either X or Y is tangent to a leaf. The restriction of R to $\Gamma(\nu) \times \Gamma(\bar{\nu}) \times \Gamma(\nu)$ will be called the transversal curvature tensor. It will be locally given by components $R^a{}_{bc\bar{d}}$. The Ricci tensor $R_{a\bar{b}} = R^c{}_{ca\bar{b}}$ will be called transversal Ricci tensor. (We would like to point out that the transversal curvature tensor coincides with the transversal part of the curvature of the second connexion introduced by I. Vaisman in [21].)

The line bundle $\wedge^n \nu$ (over M) will be called the canonical bundle of the foliation and denoted by $K(M)$. Let $\{U_\alpha\}$ be a cover of M consisting of flat local charts. The functions $f_\alpha = \det(g_{a\bar{b}})_\alpha$ on each U_α define a foliate Hermitian metric in $K(M)^*$. It is easy to show that the curvature form corresponding to the unique connexion of type (1,0) associated to this metric is locally given by

$$(7.1) \qquad \Omega = R_{a\bar{b}} dz^a \wedge \overline{dz^b},$$

where $R_{a\bar{b}}$ are the components of the transversal Ricci tensor.

Let $E \to M$ be a complex analytic foliate vector bundle with r-dimensional fibres. Denote by $(f_A^B)_{\alpha\beta}$ its transition functions. We shall denote by $\det E \to M$ the complex analytic foliate line bundle with transition functions $\varphi_{\alpha\beta} = \det(f_A^B)_{\alpha\beta}$. As in [12] one can prove

$$K(P(E^*)) = Q(E)^{-r} \otimes p^*(\det E \otimes K(M)), \tag{7.2}$$

where p denotes the projection $P(E^*) \to M$. As in the non-foliate case one can prove the following isomorphism:

$$\Omega_b^0(E) = \Omega_b^n(K(M)^* \otimes E). \tag{7.3}$$

8. Main vanishing theorem

THEOREM 8.1. *Let M be a compact complex manifold of dimension $n+m$, with a complex analytic foliation $\mathcal{F}$ of codimension n whose leaves are closed subsets. Suppose that M is endowed with a bundle-like pseudo-Kähler metric g. Suppose that $b_1(L) = 0$, where L is the generic leaf of the foliation (which is compact Hausdorff by virtue of these assumptions). Let $E \to M$ be an r-dimensional complex analytic foliate vector bundle, endowed with a foliate Hermitian metric h. Assume that E is allowable (Definition 1.2). Let (Θ_A^B) be the curvature matrix of the connexion of type $(1,0)$ associated to h and let (Ω_a^b) be the transversal curvature matrix associated to g (see §7). Assume that*

$$\begin{aligned} Q(t,X) = {} & (r+1)h_{AC}\Theta_B^C(Z_a, \overline{Z_b})X^a\overline{X^b}t^A\overline{t^B} \\ & + (R_{a\bar{b}} - \Sigma_A\, \Theta_A^A(Z_a, \overline{Z_b}))X^a\overline{X^b}h(t,t) > 0 \end{aligned} \tag{8.1}$$

for any local non-vanishing section t of E and any non-vanishing transversal vector field $X = \Sigma X^a Z_a$. (See §7 for the definition of Z_a.) Then $H^1(M, \Omega_b^0(E)) = 0$.

PROOF. We can introduce a bundle-like pseudo-Kähler metric on $P(E^*)$ in the following way. Let γ be the curvature form of the Hermitian metric (6.3) in $Q(E)$. Let g be the bundle-like pseudo-Kähler metric on M. Then $kp^*(g) + \gamma$ (where k is a sufficiently large positive constant) will be positive definite at each point by virtue of (6.4). (In fact, k is chosen such that $\gamma_{a\bar{b}} + kg_{a\bar{b}}$ be positive definite at each point.) It is obvious that this metric will be bundle-like pseudo-Kähler for the foliation introduced on $P(E^*)$ by (6.1). Hence we shall have a foliate Hermitian metric in $K(P(E^*)) \otimes Q(E)^*$. Call this metric h'. Let γ' be its curvature form. Denote by γ_1, γ_2 the curvature forms of the metrics in

$K(M)^*$ and $\det E$ respectively induced by the metric g on M and h in E. By virtue of (7.2) we shall have

$$\gamma' = -(r+1)\gamma + p^*(-\gamma_1 + \gamma_2). \tag{8.2}$$

Let us show that the expression $(\partial_a \partial_{\bar{b}} h^*_{AB}) t^A \overline{t^B}$ which appears in (6.4) can be written in a more convenient way. Recall that (6.4) was the expression of γ at a fixed point where h^* was the unit matrix and dh^* was the zero matrix. The curvature matrix of h^* is

$$\Omega^* = d''\omega^* = (d''d'h^*)h^{*-1} + (d'h^*) \wedge h^{*-1}(d''h^*)h^{*-1}. \tag{8.3}$$

At the chosen point (since $d''h^* = 0$, $h^* = I$) we shall have $\Omega^* = d''d'h^*$. E will have at this point the curvature $\Omega = -{}^t(d''d'h^*)$. Since the metric h^* is foliate, the expression $(\partial_a \partial_{\bar{b}} h^*_{AB}) t^A \overline{t^B}$ can be written $(Z_a \overline{Z}_b h^*_{AB}) t^A \overline{t^B}$ where Z_a have been introduced in §7. Then we have

$$(\partial_a \partial_{\bar{b}} h^*_{AB}) t^A \overline{t^B} = h_{AC} \Omega^C_B (Z_a, \overline{Z}_b) t^A \overline{t^B}. \tag{8.4}$$

From (6.4), (8.4), (7.1), (8.2) and (8.1), the transversal part of the matrix γ' will be negative definite. By Theorem 5.1 applied to $P(E^*)$ and to the bundle $K(P(E^*)) \otimes Q(E)^*$, there is a bundle-like pseudo-Kähler metric on $P(E^*)$ such that $H^{p,q}_b(P(E^*), K(P(E^*)) \otimes Q(E)^*) = 0$ if $p + q < n + r - 1$, where $H^{p,q}_b$ are referred to this metric. In particular $H^{0,n+r-2}_b = 0$. Using the operator $\tilde{\#}_b$ introduced in §4 one can see that

$$H^{0,n+r-2}_b(P(E^*), K(P(E^*)) \otimes Q(E)^*) \cong H^{n+r-1,1}_b(P(E^*), K(P(E^*))^* \otimes Q(E)).$$

On the other hand, by Le Potier's isomorphism and (7.3) we shall have

$$\begin{aligned} H^1(M, \Omega^0_b(E)) &\cong H^1(P(E^*), \Omega^0_b(Q(E))) \\ &\cong H^1(P(E^*), \Omega^{n+r-1}_b(K(P(E^*))^* \otimes Q(E))). \end{aligned}$$

In order to end the proof it suffices to prove that

$$\begin{aligned} &H^1(P(E^*), \Omega^{n+r-1}_b(K(P(E^*))^* \otimes Q(E))) \\ &\cong H^{n+r-1,1}_b(P(E^*), K(P(E^*))^* \otimes Q(E)). \end{aligned}$$

But this isomorphism will be a consequence of Theorem 3.1 if we prove that the assumptions of that theorem are fulfilled. Let us prove that $Q(E)$ is an allowable vector bundle. Since E is allowable, its restriction to a small saturated neighborhood will be (isomorphic to) the bundle $(L \times D \times \mathbf{C}^r)/G \to L \times_G D$, where G acts on $L \times D \times \mathbf{C}^r$ by $g(p, x, v) = (pg^{-1}, gx, \eta(g)(x)v)$. The bundle E^*

on the same neighborhood will be $(L \times D \times \mathbf{C}^r)/G \to L \times_G D$, where G acts by $g(p,x,v) = (pg^{-1},\ gx,\ \widetilde{{}^t\eta(g)(x)^{-1}}v)$. $P(E^*)$ on the same neighborhood will be $(L \times D \times P_{r-1}(\mathbf{C}))/G \to L \times_G D$, where G acts by

$$g(p,x,v) = (pg^{-1}, gx, {}^t\eta(g)(x)^{-1}v).$$

(This shows that the generic leaf of the foliation on $P(E^*)$ is also L.) Let p be the canonical projection

$$P(E^*) = (L \times D \times P_{r-1}(\mathbf{C}))/G \to L \times_G D = M.$$

The pull-back of E (on $L \times_G D$) by p will be $(L \times D \times P_{r-1}(\mathbf{C}) \times \mathbf{C}^r)/G$ where G acts by $g(p,x,\tilde{v},u) = (pg^{-1},\ gx,\ \widetilde{{}^t\eta(g)(x)^{-1}}v,\ \eta(g)(x)u)$. The subbundle S of $p^*(E)$ (see §6) will consist of those $(p,x,\tilde{v},u)$ such that $\Sigma\, v_A u^A = 0$ (we shall write $v(u) = 0$). Observe that if $v(u) = 0$, then ${}^t\eta(g)(x)^{-1}v(\eta(g)(x)u) = 0$. This shows that the quotient bundle $Q(E) = p^*(E)/S$ will be locally given by $(L \times D \times P_{r-1}(\mathbf{C}) \times \mathbf{C})/G$ where G acts by

$$g(p,x,\tilde{v},c) = (pg^{-1}, gx, \widetilde{{}^t\eta(g)(x)^{-1}}v, c).$$

This shows that $Q(E)$ is allowable. On the other hand $K(P(E^*))$ is allowable since the transversal bundle of a Hausdorff compact foliation is always allowable and the canonical bundle has been defined by means of the transversal bundle. Hence $K(P(E^*))^* \otimes Q(E)$ is allowable. We have also seen that the generic leaf of the foliation taken on $P(E^*)$ is L. Since $b_1(L) = 0$, all the assumptions of Theorem 3.1 are fulfilled. This ends the proof of our theorem.

9. Some consequences of Theorem 8.1 about the stability of complex analytic foliations

In this section we suppose that M is a compact complex manifold of dimension $n + m$ with a complex analytic foliation $\mathcal{F}$ of codimension n whose leaves are closed subsets. Let τ be the tangent bundle (of complex type $(1,0)$) and let $\nu = T(M)/\tau$ be the transversal bundle. Denote $\Psi = \Omega_b^0(\nu)$. Let Θ be the sheaf of germs of local holomorphic vector fields

$$X = X^a \frac{\partial}{\partial z^a} + X^u \frac{\partial}{\partial z^u}$$

such that the functions X^a and X^u are holomorphic and $\partial X^a / \partial z^u = 0$. Denote by Φ the sheaf of germs of holomorphic sections of τ. We shall have the following exact sequence of sheaves:

$$0 \to \Phi \to \Theta \to \Psi \to 0. \tag{9.1}$$

(See [22] and [15]. In [15] Φ, Θ and Ψ are denoted by $\bar{\Theta}_C, \hat{\Theta}_C$ and Θ_C, respectively.)

DEFINITION 9.1. (a) The foliation $\mathcal{F}$ is called infinitesimally holomorphically stable if $H^1(M, \Psi) = 0$. (This stability has been studied in [15], [5] and [6].)

(b) The foliation $\mathcal{F}$ is called Kodaira–Spencer stable if $H^1(M, \Theta) = 0$. (This stability has been studied in [17].)

REMARK. (*The meaning of the different kinds of stability introduced in Definition* 9.1). Let $\mathcal{F}$ be a complex analytic foliation on M. Let U be a small open set of some euclidean space $\mathbf{R}^s$ containing the origin. A differentiable family of deformations of $\mathcal{F}$ is a family $\{\mathcal{F}_t\}$ of complex analytic foliations on M depending differentiably on $t \in U$ such that for $t = 0$, $\mathcal{F}_0 = \mathcal{F}$. Kodaira–Spencer's stability (for $\mathcal{F}$) means that every foliation $\mathcal{F}_t$ in a small differentiable family of deformations of $\mathcal{F}$ is conjugate to $\mathcal{F}$ in the sense that there exists a biholomorphic bijective map of M sending leaves of $\mathcal{F}_t$ to leaves of $\mathcal{F}$.

Let us explain now the meaning of infinitesimal-holomorphic stability. Given two C^∞ foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on M whose local transversal manifolds are complex analytic, we shall say that $\mathcal{F}_1$ and $\mathcal{F}_2$ are transverse conjugate if there exists a diffeomorphism of M sending the leaves of $\mathcal{F}_1$ to the leaves of $\mathcal{F}_2$ and inducing a holomorphic map on each local transversal manifold. Suppose that there exists a C^∞ foliation transversal to $\mathcal{F}$. Duchamp and Kalka have proven in that case that infinitesimal-holomorphic stability means that every C^∞ foliation on M sufficiently near $\mathcal{F}$, whose local transversal manifolds are holomorphic, is transverse conjugate to $\mathcal{F}$ (theorem 3.1 of [5]). (Duchamp has communicated to me recently that he and Kalka have succeeded in removing the assumption that a C^∞ foliation transversal to $\mathcal{F}$ exists.) This gives a complete description of the meaning of infinitesimal-holomorphic stability.

From Theorem 8.1 we shall obtain the following

THEOREM 9.1. *Suppose that M is endowed with a bundle-like pseudo-Kähler metric g. Moreover, suppose that*

(a) $b_1(L) = 0$, *where L is the generic leaf,*

(b) *the transversal curvature tensor* (*see* §7) *has positive bisectional holomorphic curvature.*

Then $\mathcal{F}$ is infinitesimally holomorphically stable.

PROOF. Let us apply Theorem 8.1 to show that $H^1(M, \Omega_b^0(\nu)) = 0$. In our case g induces a foliate Hermitian metric in ν. In this case (8.1) becomes $Q(t, X) = (n+1)R(t, \bar{t}, X, \bar{X})$. Condition (b) implies $Q(t, X) > 0$ and the theorem is proven.

EXAMPLE 9.1. (Proposition 5 of [22]). Let M be the product $P_m(C) \times P_n(C)$ $(n, m > 0)$ with the natural foliation $\mathcal{F}$ of codimension n and the product metric $g = g_1 \times g_2$ where g_1 and g_2 are Fubini's metrics. The hypotheses of Theorem 9.1 are fulfilled. Then $H^1(M, \Psi) = 0$. The reasoning in [22] in order to show that $H^1(M, \Phi) = 0$ holds good. Then $\mathcal{F}$ is Kodaira–Spencer stable as a consequence of the exact cohomology sequence associated to (9.1.).

REMARK. It is not necessary to use Theorem 9.1 in this case. I. Vaisman pointed out to me the following reasoning. By [5, lemma 4.9] we have

$$H^1(P_m(\mathbf{C}) \times P_n(\mathbf{C}), \Psi) = H^0(P_n(\mathbf{C}), \Theta(P_n(\mathbf{C})) \otimes H^1(P_m(\mathbf{C}), \mathbf{C})) + H^1(P_n(\mathbf{C}), \Theta(P_n(\mathbf{C})) \otimes H^0(P_m(\mathbf{C}), \mathbf{C})) = 0,$$

where $\Theta(P_n(\mathbf{C}))$ is the sheaf of germs of holomorphic vector fields on $P_n(\mathbf{C})$.

EXAMPLE 9.2. Let V be the projective space $P_n(\mathbf{C})$. Let $p: P(T(V)) \to V$ be the projectivization of the tangent bundle $T(V)$ (of complex type (1.0)). Let M be the manifold $P(T(V))$ endowed with the foliation $\mathcal{F}$ given by the fibres of the preceding fibration. Take on M the bundle-like metric

$$g = g_{a\bar{b}} dz^a \overline{dz^b} + \Psi \tag{9.2}$$

given in the theorem on page 84 of [21], where $g_{a\bar{b}}$ is Fubini's metric on $P_n(\mathbf{C})$. This metric is pseudo-Kähler. The assumptions of Theorem 9.1 are fulfilled. Therefore $\mathcal{F}$ is infinitesimally holomorphically stable.

We are going to compute $H^1(M, \Phi)$ in order to show that $\mathcal{F}$ is also Kodaira–Spencer stable.

$T(P_n(\mathbf{C}))$ can be thought of as the set of pairs $(\langle v \rangle, w)$ with v a unitary vector of $\mathbf{C}^{n+1}$ and $w \in \langle v \rangle^\perp$. ($\langle v \rangle$ means the vector subspace generated by v.) (See [11, proposition IV, p. 217].) Therefore $P(T(P_n(\mathbf{C})))$ can be interpreted as the set of pairs $(\langle v \rangle, \langle w \rangle)$ with v, w unitary vectors of $\mathbf{C}^{n+1}$, $v \perp w$. Then

$$P(T(P_n(\mathbf{C}))) \cong U(n+1)/(U(1) \times U(1) \times U(n-1)).$$

In fact, we can identify both spaces by the following mapping:

$$\begin{aligned} U(n+1)/(U(1) \times U(1) \times U(n-1)) &\to P(T(P_n(\mathbf{C}))), \\ \tilde{\rho} &\to (\rho\langle e_1 \rangle, \rho\langle e_2 \rangle), \end{aligned} \tag{9.3}$$

where $\rho \in U(n+1)$, $\tilde{\rho}$ is its class and $\{e_1 \cdots e_{n+1}\}$ is the canonical basis of $\mathbf{C}^{n+1}$. It is easy to see that (9.3) is a well-defined isomorphism. Another way to see that this identification is possible is given by problem 10(iii) of [11, p. 463]. To abbreviate, denote

$$G = U(n+1), \qquad H = U(1)\times U(n), \qquad K = U(1)\times U(1)\times U(n-1).$$

The fibre bundle $P(T(V)) \to V$ is $G/K \to G/H$ with the natural projection (G/K is the space of cosets gK).

Let $T(G)$ be the tangent bundle of G (of complex type $(1,0)$). Let L_H be the subbundle of $T(G)$ consisting of the vectors tangent to the fibres of $G \to G/H$. Let L_K be the subbundle of $T(G)$ consisting of the vectors tangent to the fibres of $G \to G/K$. K acts on L_H and L_K on the right. $L_H/K \to G/K$ and $L_K/K \to G/K$ will be vector bundles. The fibre of L_H/K over a point $x_0 = g_0K \in G/K$ will be isomorphic to $(L_H)_{g_0}$ (the fibre over g_0). The fibre of L_K/K over x_0 will be $(L_K)_{g_0}$. Let T be the subbundle of the tangent bundle $T(G/K)$ consisting of the vectors tangent to the fibres of $G/K \to G/H$. We have a natural projection $L_H/K \to T$ whose kernel is L_H/K. Then we have the following exact sequence of vector bundles over G/K:

$$0 \to L_K/K \to L_H/K \to T \to 0. \tag{9.4}$$

If $E \to G/K$ is a holomorphic vector bundle over G/K we shall denote by $\Omega^p(E)$ the sheaf of germs of holomorphic E-valued $(p,0)$-forms on G/K. The sheaf Φ whose cohomology $H^1(M,\Phi)$ we are going to compute is $\Omega^0(T)$.

We are going to prove that L_K/K and L_H/K are trivial bundles. G acts on the latter, on the left, transforming fibres into fibres. Then $L_H/K \to G/K$ is a homogeneous vector bundle. Let us denote $\tilde{e} = eK \in G/K$ (e the unit element of G). The fibre $(L_H/K)_{\tilde{e}}$ on $\tilde{e}$ is $(L_H)_e$. Let us consider the left action of G on $(L_H)_e$ defined by the adjoint representation, $g(w) = gwg^{-1}$. Consider the right action of K on $G \times (L_H)_e$ defined by $(g,w)k = (gk, g^{-1}(w))$. One can see that $L_H/K = (G \times (L_H)_e)/K$ (see [11, p. 136]). We summarize this fact by saying that L_H/K can be thought of as the bundle associated to $G \to G/K$ with fibre $(L_H)_e$ via the adjoint representation of K on $(L_H)_e$. In this situation (see [2, p. 243]) it is known that L_H/K is trivial if there is a holomorphic map $h: G \to \mathrm{Aut}(L_H)_e$ such that $h(gk) = h(g)\,\mathrm{ad}\,k$ for every $g \in G$, $k \in K$. It suffices to take in this case $h(g) = \mathrm{ad}\,g$. This shows that L_H/K is trivial. The same reasoning shows that L_K/K is trivial too.

We are going to prove that $H^i(G/K, \Omega^0(L_H/K)) \cong H^i(G/K, \Omega^0(L_K/K)) = 0$ if $i > 0$. Since the bundles are trivial it suffices to show that $H^i(G/K, \Omega^0) = 0$, where Ω^0 is the sheaf of germs of holomorphic functions on G/K. Recall that $G/K = P(T(P_n(\mathbf{C})))$. By a theorem of Bochner–Lichnerowicz [18] (see also [8]), it suffices to show that the first Chern class of $P(T(P_n(\mathbf{C})))$ is positive definite. A representative of the first Chern class of a manifold M is given by the curvature of the connexion associated to some Hermitian metric in the dual of the canonical bundle, $K(M)^*$. We know [12] that

$$K(P(T(P_n(\mathbf{C}))))^* \cong Q(T(P_n(\mathbf{C})))^n \otimes p^*(\det T(P_n(\mathbf{C}))^* \otimes K(P_n(\mathbf{C}))^*).$$

By the same reasoning as in Theorems 9.1 and 8.1, the first Chern class of $P(T(P_n(\mathbf{C})))$ will be positive if

$$n g_{a\bar{c}} \Omega_{\bar{b}}^{\bar{c}}\left(\frac{\partial}{\partial z^d}, \frac{\partial}{\partial \bar{z}^e}\right) X^d \overline{X^e} Y^a \overline{Y^b} > 0$$

for any non-vanishing vectors X, Y tangent to $P_n(\mathbf{C})$. But this expression is just the holomorphic bisectional curvature of $P_n(\mathbf{C})$ which is >0. Therefore $H^i(G/K, \Omega^0(L_H/K)) \cong H^i(G/K, \Omega^0(L_K/K)) = 0$ if $i > 0$. From the exact cohomology sequence associated to (9.4) we get $H^1(G/K, \Omega^0(T)) \cong H^2(G/K, \Omega^0(L_K/K)) = 0$. Hence we can conclude that the *foliation $\mathcal{F}$ of Example 9.2 is Kodaira–Spencer stable.*

Remark. The foliation in the preceding example is given by a (locally trivial) fibration over a complex manifold. Nevertheless Theorem 9.1 applies to more general foliations. Hence it is natural to ask for an example of an application of Theorem 9.1 in which the foliation does not come from a fibre bundle over a complex manifold. The following example is such.

Example 9.3. Let σ be the linear automorphism of $\mathbf{C}^3$ given by the diagonal matrix

$$\begin{pmatrix} 1 & & \\ & a_1 & \\ & & a_2 \end{pmatrix}$$

where $a_1 = (-1/2) + i(\sqrt{3}/2)$, $a_2 = (-1/2) - i(\sqrt{3}/2)$. Denote also by σ the projectivity of $P_2(\mathbf{C})$ given by the preceding automorphism of $\mathbf{C}^3$. Let G be the subgroup of projectivities generated by σ. $G = \{\mathrm{id}, \sigma, \sigma^2\}$. A point of $P_2(\mathbf{C})$ will be called singular if its isotropy subgroup by the action of G is distinct from $\{\mathrm{id}\}$. There are only three singular points in $P_2(\mathbf{C})$: $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. G acts in a natural way on the tangent bundle (of type $(1,0)$) $T(P_2(\mathbf{C}))$ as well as on its projectivization $P(T(P_2(\mathbf{C})))$. Since the singular points of $P_2(\mathbf{C})$ are isolated, G

acts *freely* on $P(T(P_2(\mathbf{C})))$. $P_2(\mathbf{C})/G$ is not a manifold (because of the singular points), but a V-manifold (see [10]). Nevertheless $P(T(P_2(\mathbf{C})))/G$ is a manifold since the action of G is free. Consider the natural projection $p: P(T(P_2(\mathbf{C})))/G \to P_2(\mathbf{C})/G$. Let $\mathcal{F}$ be the foliation on $P(T(P_2(\mathbf{C})))/G$ whose leaves are the fibres of this projection. Take the metric g given by (9.2) on $P(T(P_2(\mathbf{C})))$. Let g' be the metric

$$g' = \sum_{\tau \in G} \tau^*(g).$$

Obviously g' is G-invariant, hence it induces a metric on $P(T(P_2(\mathbf{C})))/G$. This metric is bundle-like pseudo-Kähler and it is easy to see that all the assumptions of Theorem 9.1 are fulfilled. Therefore $\mathcal{F}$ is infinitesimally holomorphically stable.

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SECCIÓ DE MATEMÀTIQUES
UNIVERSITAT AUTÒNOMA DE BARCELONA
BELLATERRA, BARCELONA, SPAIN